

ON FRACTIONAL INEQUALITIES VIA MONTGOMERY IDENTITIES INTEGRALS

MEHMET ZEKI SARIKAYA*, HATICE YALDIZ, AND ERHAN SET

ABSTRACT. In the present work we give several new integral inequalities of the type Riemann-Liouville fractional integral via Montgomery identities integrals.

1. INTRODUCTION

The inequality of Ostrowski [9] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For some generalizations of this classic fact see the book [5, p.468-484] by Mitrić, Pecaric and Fink. A simple proof of this fact can be done by using the following identity [5]:

If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ with the first derivative f' integrable on $[a, b]$, then Montgomery identity holds:

$$(1.1) \quad f(t) = \frac{1}{b-a} \int_a^b f(s) ds + \int_a^b P(t, s) f'(s) ds,$$

where $P(x, t)$ is the Peano kernel defined by

$$(1.2) \quad P(t, s) := \begin{cases} \frac{s-a}{b-a}, & a \leq s < t \\ \frac{s-b}{b-a}, & t \leq s \leq b. \end{cases}$$

Suppose now that $w : [a, b] \rightarrow [0, \infty)$ is some probability density function, i.e. it is a positive integrable function satisfying $\int_a^b w(t) dt = 1$, and $W(t) = \int_a^t w(x) dx$ for $t \in [a, b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. The following identity (given by Pečarić in [8]) is the weighted generalization of the Montgomery identity:

$$(1.3) \quad f(x) = \int_a^b w(t) f(t) dt + \int_a^b P_w(x, t) f'(t) dt,$$

2000 *Mathematics Subject Classification.* 26D15, 41A55, 26D10 .

Key words and phrases. Riemann-Liouville fractional integral, Ostrowski inequality.

*corresponding author.

where the weighted Peano kernel is

$$P_w(x, t) := \begin{cases} W(t), & a \leq t < x \\ W(t) - 1, & x \leq t \leq b. \end{cases}$$

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ with $a \geq 0$ is defined by

$$(1.4) \quad \begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \\ J_a^0 f(x) &= f(x). \end{aligned}$$

Recently, many authors have studied a number of inequalities by used the Riemann-Liouville fractional integrals, see ([1]-[6], [10], [11]) and the references cited therein. More details, for necessary definitions and mathematical preliminaries of fractional calculus theory, one can consult [6], [7].

2. RESULTS

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ such that $f' \in L_p[a, b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $\alpha \geq 0$. Then, the following inequality holds:*

$$(2.1) \quad \left| \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^{\alpha+1} \int_a^b f(s) ds \right| \leq (b-a)^{\alpha+\frac{1}{q}} \left(\frac{1}{(\alpha q+1)^{\frac{1}{q}}} + \frac{1}{(q+1)^{\frac{1}{q}}} \right) \|f'\|_p.$$

Proof. We can write the Riemann-Liouville fractional integral operator as follows:

$$(2.2) \quad \Gamma(\alpha) J_a^\alpha f(b) = \int_a^b (b-t)^{\alpha-1} f(t) dt.$$

Thus, using Montgomery identity in (2.2), we have

$$(2.3) \quad \begin{aligned} \Gamma(\alpha) J_a^\alpha f(b) &= \int_a^b (b-t)^{\alpha-1} \left[\frac{1}{b-a} \int_a^b f(s) ds + \int_a^b P(t, s) f'(s) ds \right] dt \\ &= \frac{1}{b-a} \int_a^b (b-t)^{\alpha-1} \left[\int_a^b f(s) ds + \int_a^t (s-a) f'(s) ds + \int_t^b (s-b) f'(s) ds \right] dt. \end{aligned}$$

By an interchange of the order of integration, we get

$$(2.4) \quad \int_a^b (b-t)^{\alpha-1} \left(\int_a^b f(s) ds \right) dt = \frac{(b-a)^\alpha}{\alpha} \int_a^b f(s) ds,$$

$$(2.5) \quad \int_a^b (b-t)^{\alpha-1} \left(\int_a^t (s-a) f'(s) ds \right) dt = \frac{b-a}{\alpha} \int_a^b (b-s)^\alpha f'(s) ds - \frac{1}{\alpha} \int_a^b (b-s)^{\alpha+1} f'(s) ds,$$

$$(2.6) \quad \int_a^b (b-t)^{\alpha-1} \left(\int_t^b (s-b) f'(s) ds \right) dt = \frac{1}{\alpha} \int_a^b (b-s)^{\alpha+1} f'(s) ds - \frac{(b-a)^\alpha}{\alpha} \int_a^b (b-s) f'(s) ds.$$

Thus, using (2.4), (2.5) and (2.6) in (2.3) we get

$$(2.7) \quad \begin{aligned} & \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^{\alpha-1} \int_a^b f(s) ds \\ &= \int_a^b (b-s)^\alpha f'(s) ds - (b-a)^{\alpha-1} \int_a^b (b-s) f'(s) ds, \alpha \geq 0. \end{aligned}$$

By taking the modulus and applying Hölder inequality, we have

$$\begin{aligned} & \left| \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^{\alpha-1} \int_a^b f(s) ds \right| \\ & \leq \left(\int_a^b |f'(s)|^p ds \right)^{\frac{1}{p}} \left(\int_a^b (b-s)^{\alpha q} ds \right)^{\frac{1}{q}} \\ & \quad + (b-a)^{\alpha-1} \left(\int_a^b |f'(s)|^p ds \right)^{\frac{1}{p}} \left(\int_a^b (b-s)^q ds \right)^{\frac{1}{q}} \\ & = (b-a)^{\alpha+\frac{1}{q}} \left(\frac{1}{(\alpha q+1)^{\frac{1}{q}}} + \frac{1}{(q+1)^{\frac{1}{q}}} \right) \|f'\|_p. \end{aligned}$$

The proof is completed. \square

Corollary 1. *Under the assumptions Theorem 1 with $\alpha = 0$, we have*

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(s) ds \right| \leq (b-a)^{\frac{1}{q}} \left(1 + \frac{1}{(q+1)^{\frac{1}{q}}} \right) \|f'\|_p.$$

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$, for every $x \in [a, b]$ and $\alpha \geq 0$. Then the following inequality holds:*

$$(2.8) \quad \left| J_a^\alpha f(b) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)} \int_a^b f(s) ds \right| \leq \frac{M(\alpha+3)(b-a)^{\alpha+1}}{2\Gamma(\alpha+2)}.$$

Proof. By use the (2.7), we have

$$\begin{aligned}
 (2.9) \quad & \left| \Gamma(\alpha + 1) J_a^\alpha f(b) - (b-a)^{\alpha-1} \int_a^b f(s) ds \right| \\
 & \leq \int_a^b (b-s)^\alpha |f'(s)| ds + (b-a)^{\alpha-1} \int_a^b (b-s) |f'(s)| ds.
 \end{aligned}$$

Since $|f'(x)| \leq M$, we get the required inequality which the proof is completed. \square

Corollary 2. *Under the assumptions Theorem 2 with $\alpha = 0$, we have*

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(s) ds \right| \leq \frac{3(b-a)}{2} M.$$

Theorem 3. *Let $w : [a, b] \rightarrow [0, \infty)$ be a probability density function, i.e. $\int_a^b w(t) dt = 1$, and set $W(t) = \int_a^t w(x) dx$ for $a \leq t \leq b$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ such that $f' \in L_p[a, b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $\alpha \geq 0$. Then the following inequality holds:*

$$\begin{aligned}
 (2.10) \quad & \left| \Gamma(\alpha + 1) J_a^\alpha f(b) - (b-a)^\alpha \int_a^b w(s) f(s) ds \right| \\
 & \leq \|f'\|_p (b-a)^\alpha \left[\left(\int_a^b |W(s) - 1|^q ds \right)^{\frac{1}{q}} + \left(\frac{b-a}{\alpha q + 1} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

Proof. By using (1.3) in (2.2), we have

$$\begin{aligned}
 (2.11) \quad \Gamma(\alpha) J_a^\alpha f(b) &= \int_a^b (b-t)^{\alpha-1} \left[\int_a^b w(s) f(s) ds + \int_a^b P_w(t, s) f'(s) ds \right] dt \\
 &= \int_a^b (b-t)^{\alpha-1} \left(\int_a^b w(s) f(s) ds \right) dt \\
 &\quad + \int_a^b (b-t)^{\alpha-1} \left(\int_a^t W(s) f'(s) ds \right) dt \\
 &\quad + \int_a^b (b-t)^{\alpha-1} \left(\int_t^b (W(s) - 1) f'(s) ds \right) dt.
 \end{aligned}$$

By an interchange of the order of integration, we get

$$(2.12) \quad \int_a^b (b-t)^{\alpha-1} \left(\int_a^b w(s) f(s) ds \right) dt = \frac{(b-a)^\alpha}{\alpha} \int_a^b w(s) f(s) ds,$$

$$(2.13) \quad \int_a^b (b-t)^{\alpha-1} \left(\int_a^t W(s) f'(s) ds \right) dt = \frac{1}{\alpha} \int_a^b (b-s)^\alpha W(s) f'(s) ds,$$

and

$$(2.14) \quad \begin{aligned} & \int_a^b (b-t)^{\alpha-1} \left(\int_t^b (W(s) - 1) f'(s) ds \right) dt \\ &= \frac{1}{\alpha} \left[(b-a)^\alpha \int_a^b [W(s) - 1] f'(s) ds + \int_a^b (b-s)^\alpha f'(s) ds \right]. \end{aligned}$$

Thus, using (2.12), (2.13) and (2.14) in (2.11) we get

$$(2.15) \quad \begin{aligned} & \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^\alpha \int_a^b w(s) f(s) ds \\ &= (b-a)^\alpha \int_a^b [W(s) - 1] f'(s) ds + \int_a^b (b-s)^\alpha f'(s) ds. \end{aligned}$$

By taking the modulus and applying Hölder inequality, we have

$$\begin{aligned} & \left| \Gamma(\alpha+1) J_a^\alpha f(b) - (b-a)^\alpha \int_a^b w(s) f(s) ds \right| \\ & \leq (b-a)^\alpha \left(\int_a^b |f'(s)|^p ds \right)^{\frac{1}{p}} \left(\int_a^b |W(s) - 1|^q ds \right)^{\frac{1}{q}} \\ & \quad + \left(\int_a^b |f'(s)|^p ds \right)^{\frac{1}{p}} \left(\int_a^b (b-s)^{\alpha q} ds \right)^{\frac{1}{q}} \\ & = \|f'\|_p (b-a)^\alpha \left[\left(\int_a^b |W(s) - 1|^q ds \right)^{\frac{1}{q}} + \left(\frac{b-a}{\alpha q + 1} \right)^{\frac{1}{q}} \right] \end{aligned}$$

which the proof is completed. \square

Corollary 3. *Under the assumptions Theorem 3 with $\alpha = 0$, we have*

$$\left| f(b) - \int_a^b w(s) f(s) ds \right| \leq \left[\left(\int_a^b |W(s) - 1|^q ds \right)^{\frac{1}{q}} + (b-a)^{\frac{1}{q}} \right] \|f'\|_p.$$

Theorem 4. *Let $w : [a, b] \rightarrow [0, \infty)$ be a probability density function, i.e. $\int_a^b w(t) dt = 1$, and set $W(t) = \int_a^t w(x) dx$ for $a \leq t \leq b$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for*

$t > b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$, for every $x \in [a, b]$ and $\alpha \geq 0$. Then the following inequality holds:

$$(2.16) \quad \left| \Gamma(\alpha + 1) J_a^\alpha f(b) - (b - a)^\alpha \int_a^b w(s) f(s) ds \right| \leq M (b - a)^\alpha \left(\int_a^b |W(s) - 1| ds - \frac{b - a}{\alpha + 1} \right).$$

Proof. From (2.15), we have

$$(2.17) \quad \begin{aligned} & \left| \Gamma(\alpha + 1) J_a^\alpha f(b) - (b - a)^\alpha \int_a^b w(s) f(s) ds \right| \\ & \leq (b - a)^\alpha \int_a^b |W(s) - 1| |f'(s)| ds + \int_a^b (b - s)^\alpha |f'(s)| ds. \end{aligned}$$

By using $|f'(x)| \leq M$, the proof is completed. \square

REFERENCES

- [1] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Inequal. in Pure and Appl. Math, 10(3), 2009, Art. 97, 6 pp.
- [2] Z. Dahmani, L. Tabharit and S. Taf, *Some fractional integral inequalities*, Nonlinear Science Letters A, 2(1), 2010, p.155-160.
- [3] Z. Dahmani, L. Tabharit and S. Taf, *New inequalities via Riemann-Liouville fractional integration*, J. Advance Research Sci. Comput., 2(1), 2010, p.40-45.
- [4] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differentiable equations of fractional order*, Springer Verlag, Wien, 1997, p.223-276.
- [5] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.
- [6] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differentiable equations of fractional order*, Springer Verlag, Wien, 1997, p.223-276.
- [7] S. G. Samko, A. A Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives Theory and Application*, Gordon and Breach Science, New York, 1993.
- [8] J. Pečarić, *On the Čebyšev inequality*, Bul. Inst. Politehn. Timisoara, 25(39), 1980, pp.10-11.
- [9] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. 10(1938), 226-227.
- [10] M.Z. Sarikaya and H. Ogunmez, *On new inequalities via Riemann-Liouville fractional integration*, arXiv:1005.1167v1, submitted.
- [11] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, DÜZCE, TURKEY

E-mail address: sarikayamz@gmail.com

E-mail address: yaldizhatice@gmail.com

E-mail address: erhanset@yahoo.com